Fitting the Cusp Catastrophe Model

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1. Introduction

Catastrophe theory describes how small, continuous changes in control parameters (i.e., independent variables that influence the state of a system) can have sudden, discontinuous effects on dependent variables. Such discontinuous, jump-like changes are called phase-transitions or catastrophes. Examples include the sudden collapse of a bridge under slowly mounting pressure, and the freezing of water when temperature is gradually decreased. Catastrophe theory was developed and popularized in the early 1970's (Thom, 1975; Zeeman, 1977). After a period of criticism (Zahler & Sussmann, 1977) catastrophe theory is now well established and widely applied, for instance in the field of physics, (e.g., Aerts et al., 2003; Tamaki, Torii, & Meada, 2003), chemistry (e.g., Wales, 2001), biology (e.g., Torres, 2001; van Harten, 2000), and in the social sciences (e.g., Hołyst, Kacperski, & Schweitzer, 2000).

In psychology, catastrophe theory has been applied to multi-stable perception (Stewart & Peregoy, 1983), transitions between Piagetian stages of child development (van der Maas & Molenaar, 1992), the perception of apparent motion (Ploeger, van der Maas, & Hartelman, 2002), sudden transitions in attitudes (van der Maas, Kolstein, & van der Pligt, 2003), and motor learning (Newell, Liu, & Mayer-Kress, 2001; Wimmers, Savelsbergh, van der Kamp, & Hartelman, 1998), to name just a few. Before proceeding to describe the statistical method required to fit the most popular catastrophe model – the cusp model – we first outline the core principles of catastrophe theory (for details see Gilmore, 1981; Poston & Stewart, 1978).

2. Catastrophe Theory

A key idea in catastrophe theory is that the system under study is driven toward an equilibrium state. This is best illustrated by imagining the movement of a ball on a curved one-dimensional surface, as in Figure 1. The ball represents the state of the system, whereas gravity represents the driving force.

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Insert Figure 1 about here

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Figure 1, middle panel, displays three possible equilibria. Two of these states are stable states (i.e., the valleys or minima) – when perturbed, the behavior of the system will remain relatively unaffected. One state is unstable (i.e., a hill or maximum) – only a small perturbation is needed to drive the system toward a different state.

Systems that are driven toward equilibrium values, such as the little ball in Figure 1, may be classified according to their configuration of critical points, that is, points at which the first or possibly second derivative equals zero. When the configuration of critical points changes, so does the qualitative behavior of the system. For instance, Figure 1 demonstrates how the local minimum (i.e., a critical point) that contains the little ball suddenly disappears as a result of a gradual change in the surface. As a result of this gradual change, the ball will suddenly move from its old position to a new minimum. These ideas may be quantified by postulating that the state of the system, x, will change over time t according to

$$dx/dt = -dV(x;c)/dx\,, \tag{1}$$

where $V(x;c)$ is the potential function that incorporates the control variables c that affect the state of the system. $V(x;c)$ yields a scalar for each state x and vector of control variables c. The concept of a potential function is very general – for instance, a potential function that is quadratic in x will yield the ubiquitous normal distribution. A system whose dynamics obey Eq. 1 is said to be a gradient dynamical system. When the right-hand side of Eq. 1 equals zero, the system is in equilibrium.

As the behavior of catastrophe models can become extremely complex when the number of behavioral and control parameters is increased, we will focus here on the simplest and most often applied catastrophe model that shows discontinuous behavior: the cusp model. The cusp model consists of one *behavioral* variable and only two *control* variables. This may seem like a small number, especially since there are probably numerous variables that exert some kind of influence on a real-life system – however, very few of these are likely to qualitatively affect transitional behavior. As will be apparent soon, two control variables already allow for the prediction of quite intricate transitional behavior. The potential function that goes with the cusp model is $V(x;c) = -\frac{1}{4}x^4 + \frac{1}{2}bx^2 + ax$, where *a* and *b* are the control variables. Figure 2 summarizes the behavior of the cusp model by showing, for all values of the control variables, those values of the behavioral variable for which the system is at equilibrium. That is, Figure 2 shows the states for which the derivative of the potential function is zero (i.e., $V'(x;c) = -x^3 + bx + a = 0$). Note that one entire panel from Figure 1 is associated with only one (i.e., a minimum), or three (i.e., two minima and one maximum) points on the cusp surface in Figure 2.

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Insert Figure 2 about here

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We now discuss some of the defining characteristics of the cusp model in terms of a model for attitudinal change (Flay, 1978; van der Maas et al., 2003; Zeeman, 1977). More specifically, we will measure attitude as regards political preference, ranging from left-wing to right-wing. Two control variables that are important for attitudinal change are *involvement* and *information*. The most distinguishing behavior of the cusp model takes places in the foreground of Figure 2, for the highest levels of involvement. Assume that the lower sheet of the cusp surface corresponds to equilibrium states of being left-wing. As “information” (e.g., experience or environmental effects) more and more favors a right-wing view, not much change will be apparent at first, but at some level of information, a sudden jump to the upper, "right-wing" sheet occurs. When subsequent information becomes available that favors the left-wing view, the system eventually jumps back from the upper sheet unto the lower sheet – but note that this jump does not occur at the same position! The system needs additional impetus to change from one state to the other, and this phenomenon is called *hysteresis*.

Figure 2 also shows that a gradual change of political attitude is possible, but only for low levels of involvement (i.e., in the background of the cusp surface). Now assume one's political attitude starts out at the neutral point in the middle of the cusp surface, and involvement is increased. According to the cusp model, an increase in involvement will lead to polarization, as one has to move either to the upper sheet or to the lower sheet (i.e., *divergence*), because for high levels of involvement, the intermediate position is *inaccessible*. Hysteresis, divergence, and inaccessibility are three of eight *catastrophe flags* (Gilmore, 1981), that is, qualitative properties of catastrophe models. Consequently, one method of investigation is to look for the catastrophe flags (i.e., catastrophe detection).

A major challenge in the search of an adequate cusp model is the definition of the control variables. In the cusp model, the variable that causes divergence is called the splitting variable (i.e., involvement), and the variable that causes hysteresis is called the normal variable (i.e., information). When the normal and splitting variable are correctly identified, and the underlying system dynamics are given by catastrophe theory, this often provides surprisingly elegant insights that cannot be obtained from simple linear models. In the following, we will ignore both the creative aspects of defining appropriate control variables and the qualitative testing of the cusp model using catastrophe flags (van der Maas & Molenaar, 1992). Instead, we will focus on the problem of statistically fitting a catastrophe model to empirical data.

3. Fitting the Cusp Catastrophe Model to Data

Several cusp fitting procedures have been proposed, but none is completely satisfactory (for an overview see Hartelman, 1997; van der Maas et al., 2003). The most important obstacle is that the cusp equilibrium surface is cubic in the dependent variable. This means that for control variables located in the bifurcation area (cf. Figure 2, bottom panel), two values of the dependent variable are plausible (i.e., left-wing/lower sheet and right-wing/upper sheet), whereas one value, corresponding to the unstable intermediate state, is definitely not plausible. Thus, it is important to distinguish between minima of the potential function (i.e., stable states) and maxima of the potential function (i.e., unstable states).

Two methods for fitting the cusp catastrophe models, namely GEMCAT I and II (Lange, Oliva, & McDade, 2000; Oliva, DeSarbo, Day, & Jedidi, 1987) and Guastello's polynomial regression technique (Guastello, 1988, 1992) both suffer from the fact that they consider as the starting point for statistical fitting only those values for the derivative of the potential function that equal zero. The equation $dx/dt = -dV(x;c)/dx = -x^3 + bx + a = 0$ is, however, valid both for minima and maxima – hence, neither GEMCAT nor the polynomial regression technique are able to distinguish between stable equilibria (i.e., minima) and unstable equilibria (i.e., maxima). Obviously, the distinction between stable and unstable states is very important when fitting the cusp model, and neglecting this distinction renders the above methods suspect (for a more detailed critique on the GEMCAT and polynomial regression techniques see Alexander, Herbert, Deshon, & Hanges, 1992; van der Maas et al., 2003).

The most principled method for fitting catastrophe models, and the one under discussion here, is the maximum likelihood method developed by Cobb and co-workers (Cobb 1978, 1981, Cobb & Zacks, 1985). First, Cobb proposed to make catastrophe theory stochastic by adding a Gaussian white noise driving term $dW(t)/dt$ with standard deviation $D(x)$ to the potential function, leading to

$$dx/dt = -dV(x;c)/dx + D(x)\,dW(t)/dt. \quad (2)$$

Eq. 2 is a stochastic differential equation (SDE) , in which the deterministic term on the right-hand side, $-dV(x;c)/dx$, is called the (instantaneous) drift function, while $D^2(x)$ is called the (instantaneous) diffusion function, and $W(t)$ is a Wiener process (i.e., idealized Brownian motion). The function $D^2(x)$ is the infinitesimal variance function and determines the relative influence of the noise process (for details on SDE's see Gardiner, 1983; Honerkamp, 1994).

Under the assumption of additive noise (i.e., $D(x)$ is a constant and does not depend on *x*), it can be shown that the modes (i.e., local maxima) of the empirical probability density function (pdf) correspond to stable equilibria, whereas the antimodes of the pdf (i.e., local minima) correspond to unstable equilibria (see e.g., Honerkamp, 1994, p. 273). More generally, there is a simple one-to-one correspondence between an additive noise SDE and its stationary pdf. Hence, instead of fitting the drift function of the cusp model directly, it can also be determined by fitting the pdf:

$$p(y \mid \alpha, \beta) = N \exp\left[-\tfrac{1}{4}y^4 + \tfrac{1}{2}\beta y^2 + \alpha y\right], \quad (3)$$

where *N* is a normalizing constant. In Eq. 3, the observed dependent variable *x* has been rescaled by $y = (x - \lambda)/\sigma$, and α and β are linear functions of the two control variables *a* and *b* as follows: $\alpha = k_0 + k_1 a + k_2 b$ and $\beta = l_0 + l_1 a + l_2 b$. The parameters λ, σ, k_0, k_1, k_2, l_0, l_1, and l_2 can be estimated using maximum likelihood procedures (Cobb & Watson, 1980).

Although the maximum likelihood method of Cobb is the most elegant and statistically satisfactory method for fitting the cusp catastrophe model to date, it is not used often. One reason may be that Cobb's computer program for fitting the cusp model can sometimes behave erratically. This problem was addressed in (Hartelman, 1997; Hartelman, van der Maas, & Molenaar, 1998), who outlined a more robust and more flexible version of Cobb's original program. The improved program, Cuspfit, uses a more reliable optimization routine, allows the user to constrain parameter values and to employ different sets of starting values, and is able to fit competing models such as the logistic model. Cuspfit is available at *http://users.fmg.uva.nl/hvandermaas/*.

We now illustrate Cobb's maximum likelihood procedure with a practical example on sudden transitions in attitudes (van der Maas et al., 2003). The data set used here is taken from Stouffer et al. (1950), and has been discussed in relation to the cusp model in Latané and Nowak (1994). US soldiers were asked their opinion about three issues (i.e., post-war conscription, demobilization, and the Women's Army Corps). An individual attitude score was obtained by combining responses on different questions relating to the same issue, resulting in an attitude score that could vary between 0 (unfavorable) to 6 (favorable). In addition, respondents indicated the intensity of their opinion on a six-point scale. Thus, this data set consists of one behavioral variable (i.e., attitude) and only one control variable (i.e., the splitting variable 'intensity').

Figure 3 displays the histograms of attitude scores for each level of intensity separately. The data show that as intensity increases, the attitudes become polarized (i.e., divergence) resulting in a bimodal histogram for the highest intensities. The dotted line shows the fit of the cusp model. The maximum likelihood method as implemented in Cuspfit allows for easy model comparison. For instance, one popular model selection method is the Bayesian information criterion (BIC; e.g., Raftery, 1995), defined as $BIC = -2 \log L + k \log n$, where L is the maximum likelihood, k is the number of free parameters, and n is the number of observations. The BIC implements Occam's razor by quantifying the trade-off between goodness-of-fit and parsimony, models with lower BIC values being preferable.

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Insert Figure 3 about here

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The cusp model, whose fit is shown in Figure 3, has a BIC value of 1787. The Cuspfit program is also able to fit competing models to the data. An example of these is the logistic model, which allows for rapid changes in the dependent variable but cannot handle divergence. The BIC for the logistic model was 1970. To get a feeling for how big this difference really is, one may approximate $P(\text{logistic} \mid \text{data})$, the probability that the logistic model is true and the cusp model is not, given the data, by $\exp\left\{-\tfrac{1}{2}\left(BIC_{\text{logistic}}\right)\right\} \Big/ \left[\exp\left\{-\tfrac{1}{2}\left(BIC_{\text{logistic}}\right)\right\} + \exp\left\{-\tfrac{1}{2}\left(BIC_{\text{cusp}}\right)\right\}\right]$ (e.g., Raftery, 1995). This approximation estimates $P(\text{logistic} \mid \text{data})$ to be about zero – consequently, the complement $P(\text{cusp} \mid \text{data})$ equals about one.

One problem of the Cobb method remaining to be solved is that the convenient relation between the pdf and the SDE (i.e., modes corresponding to stable states, antimodes corresponding to unstable states) breaks down when the noise is multiplicative, that is, when $D(x)$ in Eq. 2 depends on *x*. Multiplicative noise is often believed to be present in economic and financial systems (e.g., time series of short-term interest rates, Jiang & Knight, 1997). In general, multiplicative noise arises under nonlinear transformations of the dependent variable *x*. In contrast, deterministic catastrophe theory is invariant under any smooth and revertible transformation of the dependent variable. Thus, Cobb's stochastic catastrophe theory loses some of the generality of its deterministic counterpart (see Hartelman, 1997, for an in-depth discussion of this point).

4. Summary and Recommendation

Catastrophe theory is a theory of great generality that can provide useful insights as to how behavior may radically change as a result of smoothly varying control variables. We discussed three statistical procedures for fitting one of the most popular catastrophe models, i.e., the cusp model. Two of these procedures, Guastello's polynomial regression technique and GEMCAT, are suspect because these methods are unable to distinguish between stable and unstable equilibria. The maximum likelihood method developed by Cobb does not have this problem. The one remaining problem with the method of Cobb is that it is not robust to nonlinear transformations of the dependent variable. Future work, along the lines of Hartelman (1997), will have to find a solution to this challenging problem.

5. Bibliography

Aerts, D., Czachor, M., Gabora, L., Kuna, M., Posiewnik, A., Pykacz, J., & Syty, M. (2003). Quantum morphogenesis: A variation on Thom's catastrophe theory. *Physical Review E*, 67, 51926-1-13.

Alexander, R. A., Herbert, G. R., Deshon, R. P., & Hanges, P. J. (1992). An examination of least squares regression modeling of catastrophe theory. *Psychological Bulletin, 111*, 366-374.

Cobb, L. (1978). Stochastic catastrophe models and multimodal distributions. *Behavioral Science, 23*, 360-374.

Cobb, L. (1981). Parameter estimation for the cusp catastrophe model. *Behavioral Science, 26*, 75-78.

Cobb, L., & Watson, B. (1980). Statistical catastrophe theory: An overview. *Mathematical Modelling, 1*, 311-317.

Cobb, L., & Zacks, S. (1985). Applications of catastrophe theory for statistical modeling in the biosciences. *Journal of the American Statistical Association, 80*, 793-802.

Flay, B. R. (1978). Catastrophe theory in social psychology: Some applications to attitudes and social behavior. *Behavioral Science, 23*, 335-350.

Gardiner, C. W. (1983). *Handbook of stochastic methods*. Berlin: Springer-Verlag.

Gilmore, R. (1981). *Catastrophe theory for scientists and engineers*. New York: Dover.

Guastello, S. J. (1988). Catastrophe modeling of the accident process: Organizational subunit size. *Psychological Bulletin, 103*, 246-255.

Guastello, S. J. (1992). Clash of the paradigms: A critique of an examination of the polynomial regression technique for evaluating catastrophe theory hypotheses. *Psychological Bulletin, 111*, 375-379.

Hartelman, P. (1997). *Stochastic catastrophe theory*. Unpublished doctoral dissertation.

Hartelman, P. A. I., van der Maas, H. L. J., & Molenaar, P. C. M. (1998). Detecting and modeling developmental transitions. *British Journal of Developmental Psychology, 16*, 97-122.

Hołyst, J. A., Kacperski, K., & Schweitzer, F. (2000). Phase transitions in social impact models of opinion formation. *Physica A, 285*, 199-210.

Honerkamp, J. (1994). *Stochastic dynamical systems*. New York: VCH Publishers.

Jiang, G. J., & Knight, J. L. (1997). A nonparametric approach to the estimation of diffusion processes, with an application to a short-term interest rate model. *Econometric Theory, 13*, 615-645.

Lange, R., Oliva, T. A., & McDade, S. R. (2000). An algorithm for estimating multivariate catastrophe models: GEMCAT II. *Studies in Nonlinear Dynamics and Econometrics, 4*, 137-168.

Latané, B., & Nowak, A. (1994). Attitudes as catastrophes: From dimensions to categories with increasing involvement. In R. R. Vallacher & A. Nowak (Eds.), *Dynamical systems in social psychology*, pp. 219-249. San Diego, CA: Academic Press.

Newell, K. M., Liu, Y.-T., & Mayer-Kress, G. (2001). Time scales in motor learning and development. *Psychological Review*, 108, 57-82.

Oliva, T., DeSarbo, W., Day, D., & Jedidi, K. (1987). GEMCAT: A general multivariate methodology for estimating catastrophe models. *Behavioral Science, 32*, 121-137.

Ploeger, A., van der Maas, H. L. J., & Hartelman, P. A. I. (2002). Stochastic catastrophe analysis of switches in the perception of apparent motion. *Psychonomic Bulletin & Review*, 9, 26-42.

Poston, T., & Stewart, I. (1978). *Catastrophe theory and its applications*. New York: Dover.

Raftery, A. E. (1995). Bayesian model selection in social research. *Sociological Methodology, 25*, 111-163.

Stewart, I. N., & Peregoy, P. L. (1983). Catastrophe theory modeling in psychology. *Psychological Bulletin*, 94, 336-362.

Stouffer, S. A., Guttman, L., Suchman, E. A., Lazarsfeld, P. F., Star, S. A., & Clausen, J. A. (1950). *Measurement and prediction*. Princeton, NJ: Princeton University Press.

Tamaki, T., Torii, T., & Meada, K. (2003). Stability analysis of black holes via a catastrophe theory and black hole thermodynamics in generalized theories of gravity. *Physical Review D*, 68, 24028-1-9.

Thom, R. (1975). *Structural stability and morphogenesis*. New York: Benjamin-Addison Wesley.

Torres, J.-L. (2001). Biological power laws and Darwin's principle. *Journal of Theoretical Biology, 209*, 223-232.

van der Maas, H. L. J., & Molenaar, P. C. M. (1992). Stagewise cognitive development: An application of catastrophe theory. *Psychological Review*, 99, 395-417.

van der Maas, H. L. J., Kolstein, R., & van der Pligt, J. (2003). Sudden transitions in attitudes. *Sociological Methods and Research, 32*, 125-152.

van Harten, D. (2000). Variable noding in *Cyprideis torosa* (Ostracoda, Crustacea): An overview, experimental results and a model from catastrophe theory. *Hydrobiologica, 419*, 131-139.

Wales, D. J. (2001). A microscopic basis for the global appearance of energy landscapes. *Science, 293*, 2067-2070.

Wimmers, R. H., Savelsbergh, G. J. P., van der Kamp, J., & Hartelman, P. (1998). A developmental transition in prehension modeled as a cusp catastrophe. *Developmental Psychobiology, 32*, 23-35.

Zahler, R. S., & Sussmann, H. J. (1977). Claims and accomplishments of applied catastrophe theory. *Nature, 269*, 759-763.

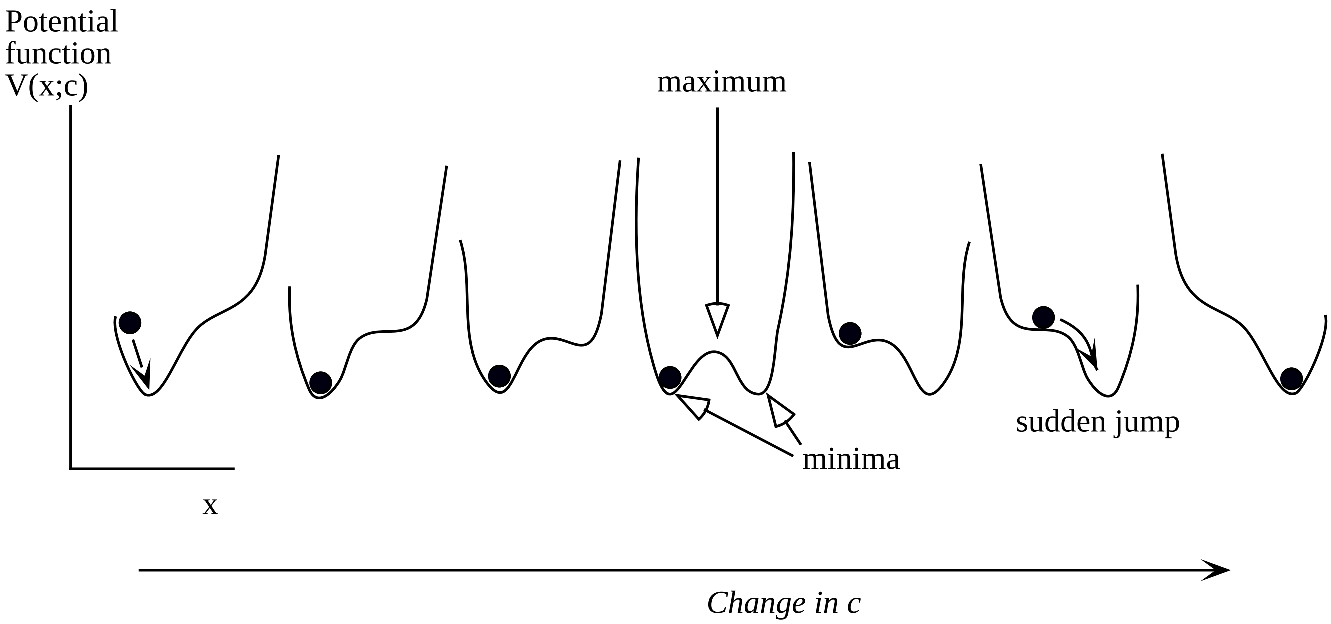
Zeeman, E. C. (1977). *Catastrophe theory: Selected papers (1972-1977)*. New York: Addison-Wesley.

Figure captions

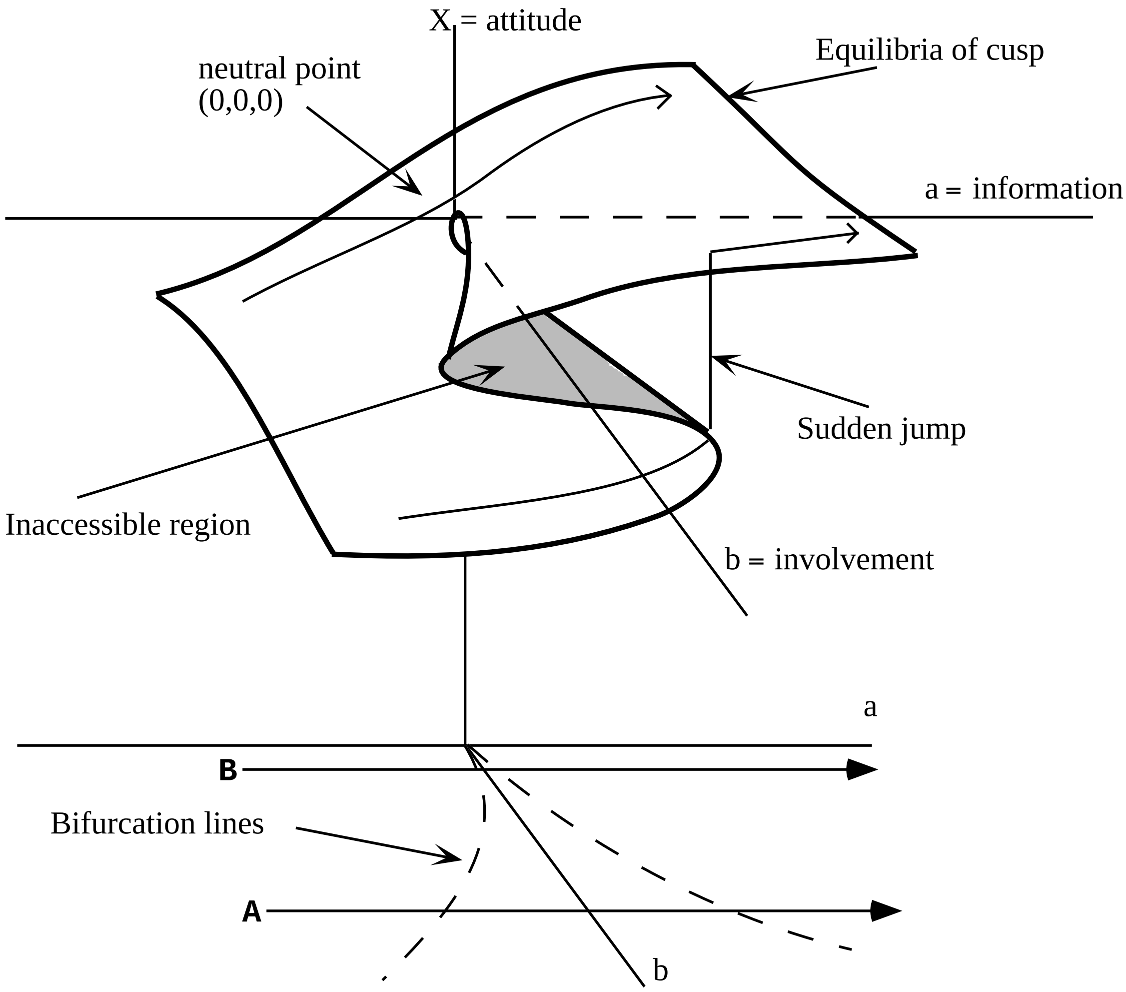
Figure 1. Smooth changes in a potential function may lead to a sudden jump. $V(x;c)$ is the potential function, and *c* denotes the set of control variables.

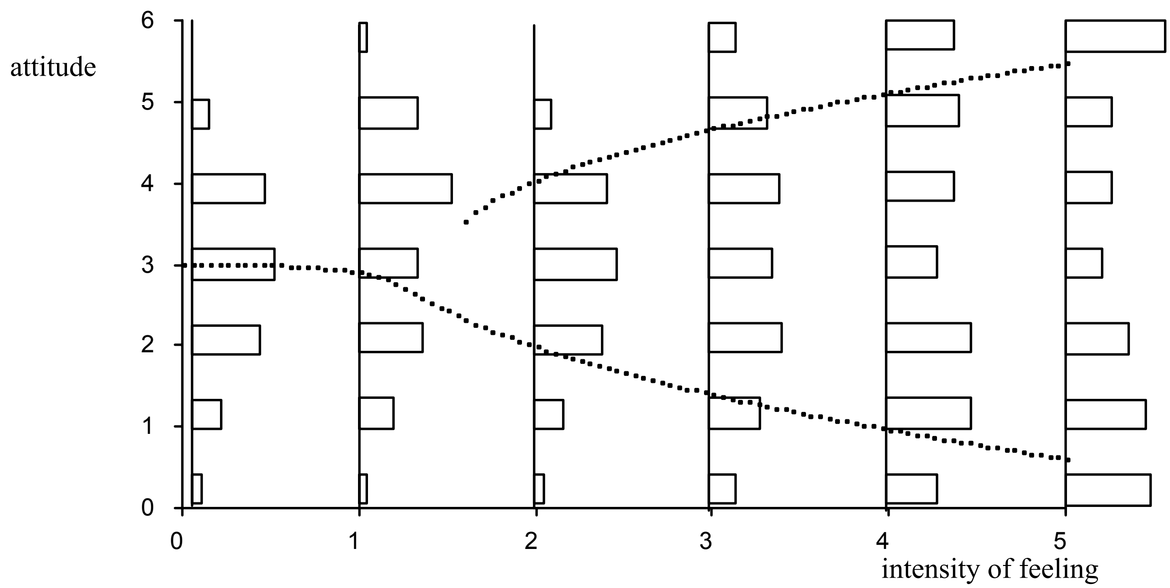
Figure 2. The cusp catastrophe model for attitude change. Of the two control variables, ‘information’ is the *normal* variable, and ‘involvement’ is the *splitting* variable. The behavioral variable is ‘attitude’. The lower panel is a projection of the ‘bifurcation’ area onto the control parameter plane. The bifurcation set consists of those values for ‘information’ and ‘involvement’ combined that allow for more than one attitude. See text for details. Adapted from van der Maas et al. (2003).

Figure 3. Histograms of attitude scores for five intensities of feeling (data from Stouffer et al., 1950). The dotted line indicates the fit of the cusp model. Both the data and the model show that bimodality in attitudes increases with intensity of feeling. Adapted from van der Maas et al. (2003).



Potential
function
V(x;c)
maximum
minima
sudden jump
x
Change in c


X = attitude
neutral point
(0,0,0)
Equilibria of cusp
a = information
Sudden jump
Inaccessible region
b = involvement
a
B
Bifurcation lines
A
b


attitude
6
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intensity of feeling